January 1993 General Examination in Analysis (administered by J. B. Brown).

Work at least 8 problems.

1.a) Define what it means to say that a subset M of [0,1] is (i) nowhere dense, (ii) first category, (iii) Lebesgue measurable.

b) Give an example of a first category subset of [0,1] of measure 1.

2. Given a measure space $(\Omega, \mathbf{A}, \mu)$ and a function $f\colon \Omega \to R$, (a) define what it means to say that f is $\mathbf{A}$-measurable. (b) Prove that if f and g are $\mathbf{A}$-measurable, then $f + g$ is $\mathbf{A}$-measurable.

(Hypothesis for 3-6) Let $f, f_1, f_2, \ldots$ be real valued functions which are measurable with respect to a σ-algebra $\mathbf{A}$ on a set Ω, and let μ be a (finite) measure on $\mathbf{A}$.

3. Define what it means to say that (a) $\{f_n\}$ converges to f in meansure (μ), (b) $\{f_n\}$ converges to f uniformly, (c) $\{f_n\}$ converges to f almost everywhere (μ), (d) $\{f_n\}$ converges to f in the $L^1(\mu)$ sense, (e) $\{f_n\}$ converges to f pointwise.

4. Line up the notions of convergence of #3 in-so-far-as which implies which. Give an example which shows that at least two of these implications don't hold if the measure μ is σ-finite rather than finite.

5. Prove that if $\{f_n\}$ converges pointwise on Ω to some function g, then g is $\mathbf{A}$-measureable.

6. Give an example where $\{f_n\}$ converges in measure (μ) to a function f but (f_n) does not converge almost everywhere (μ) to f.

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7. State the "Lebesgue Dominated Convergence Theorem" (about moving "$\lim_{n\to\infty}$" inside or outside the integral sign).

8. Define what it means to say that a function $f\colon [0,1] \to R$ is absolutely continuous, and give an example of a continuous function f which is of bounded variation on [0,1] but not absolutely continuous.

9. Define $\ell^p, L^p[0,1], L^p(R)$, and $L^p(\mu)$ for $0 < p \le \infty$ { you can make the L^p-spaces collections of functions or collections of equivalence classes of functions, either way is OK }.

10. (a.) Prove that $L^2[0,1] \subseteq L^1[0,1]$.

(b.) Give an example to show that $L^2(R) \not\subseteq L^1(R)$.